Papakyriakopoulos and 3-Dimensional Topology

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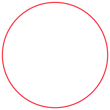
December 2014

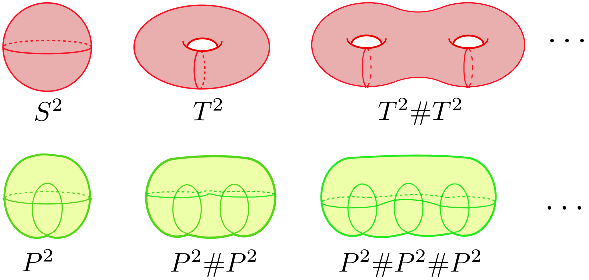
n-**manifold**: compact, connected, metric space, locally $\cong \mathbb{R}^n$

E.g. $S^n = n\text{-sphere} = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ $(n \geq 1)$

$n = 0:$ 

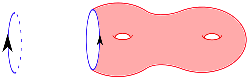
$n = 1:$  S^1

$n = 2:$



Homology $H_1(X) =$ abelian group

$= \{\text{loops in } X\}/\{\text{loops that bound}\}$



F, F' 2-manifolds. Then

$$F \cong F' \iff H_1(F) \cong H_1(F')$$

Hence

(1) $H_1(F) = 0 \iff F \cong S^2$

(2) Homeomorphism problem for 2-manifolds algorithmically solvable

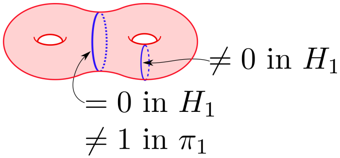
$n \geq 4$: Homeomorphism problem for n-manifolds unsolvable

(Markov, 1958)

- $n = 3$: Fundamental group $\pi_1(X)$ (Poincaré, 1895)

  $= \{\text{loops in } X\}/\{\text{homotopy = continuous deformation}\}$

$$H_1(X) = \pi_1(X)^{\mathrm{ab}}$$



There exists a 3-manifold M with $H_1(M) = 0$ but $\pi_1(M) \neq 1$

So $M \not\cong S^3$ (Poincaré, 1904)

Poincaré Conjecture $\pi_1(M) = 1 \iff M \cong S^3$

Proved by Perelman (2003)

Henri Poincaré (1854–1912)

CINQUIÈME COMPLÉMENT À L'ANALYSIS SITUS.

Par M. H. Poincaré, à Paris.

Il resterait une question à traiter :

Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?

En d'autres termes, peut-on tracer les cycles K_1'' et K_2'' de telle façon qu'ils ne soient pas bouclés et ne se coupent pas; que les équivalences

$$K_1' \equiv K_2' \equiv 0, \quad K_1'' \equiv K_2'' \equiv 0$$

entraînent les équivalences

$$C_1 \equiv C_2 \equiv C_3 \equiv C_4 \equiv 0$$

et que cependant la surface W ne puisse pas être regardée comme homéomorphe à elle-même de telle façon qu'aux cycles C_1, C_2, C_3, C_4 correspondent les cycles C_1', C_2', C_3', C_4'; que les équivalences

$$K_1' \equiv K_2' \equiv 0$$

entraînent $C_1' \equiv C_3' \equiv 0$ et réciproquement; et qu'enfin les équivalences

$$K_1'' \equiv K_2'' \equiv 0$$

entraînent $C_2' \equiv C_4' \equiv 0$ et réciproquement?

Mais cette question nous entraînerait trop loin.

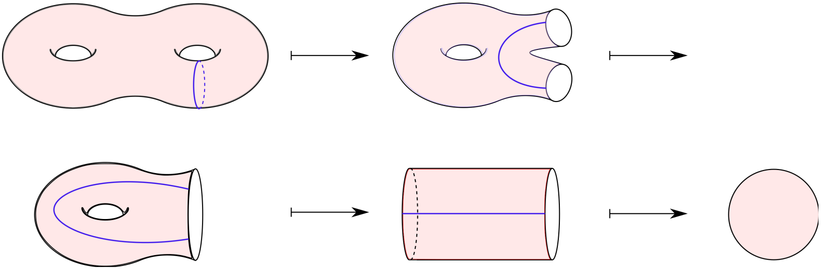
Paris, 3 novembre 1903.

H. Poincaré.


НЕТ!

Embedded $(n-1)$-manifolds $\subset$ n−manifolds are useful

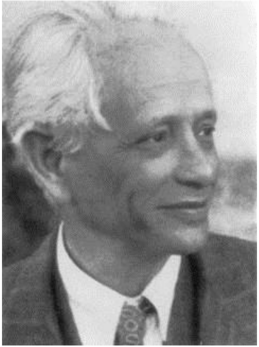
e.g. can cut along them:



Want useful 2-manifolds $\subset$ 3-manifolds

Dehn's Lemma (1910)

"In a 3-manifold, existence of a singular disk implies existence of an embedded disk"



Kneser (1929) found mistake in proof

Finally proved by Christos Papakyriakopoulos (1957)

§ 3.

Ein topologischer Hilfssatz (das „Lemma").

Wir werden des öfteren uns des folgenden Satzes aus der Topologie der Flächenkomplexe bedienen müssen, den wir wegen seiner ausgezeichneten Stellung in dieser Arbeit kurz das *Lemma* nennen wollen.

Ein Flächenkomplex C_2 *möge ganz im Inneren einer homogenen Mannigfaltigkeit* M_n $(n > 2)$ *liegen. Auf dem* C_2 *möge die Kurve* k *ein Elementarflächenstück* E_2' *begrenzen. Hat* E_2' *auf seinem Rande keine Singularitäten, dann begrenzt* k *in der* M_n *auch ein völlig singularitätenfreies Elementarflächenstück.*

Dieser Satz ist selbstverständlich, wenn die M_n mehr als drei Dimensionen hat, denn in einer solchen Mannigfaltigkeit ist jedem zweidimensionalen Gebilde ein homöomorphes ohne Singularitäten benachbart. Wir können also $n = 3$ annehmen.

ANNALS OF MATHEMATICS
Vol. 66, No. 1, July, 1957
Printed in U.S.A.

ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS

BY C. D. PAPAKYRIAKOPOULOS

(Received February 28, 1957)

Dedicated to Professor N. Kritikos

§1. Introduction

The present paper contains a proof of *Dehn's lemma* and an analogous result that we call the *sphere theorem*, from which other theorems follow.[1]

"Although it is not yet possible to evaluate the impact of this important paper on the development of geometric topology, it has already led to renewed attack on the problem of classifying the 3-dimensional manifolds; significant results have been and are being obtained. A complete solution has suddenly become a definite possibility."

(R. H. Fox)

The perfidious lemma of Dehn
Drove many a good man insane.
But Christos D. Pap-
akyriakop-
oulos proved it without any strain.

(J. Milnor)

Papakyriakopoulos had earlier proved the Loop Theorem (1957)

Combining Loop Theorem and Dehn's Lemma gives

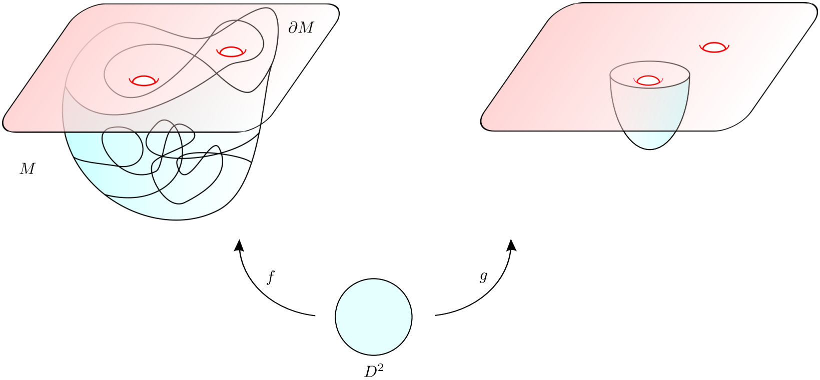
Disk Theorem

M a 3-manifold with boundary ∂M. If there exists a map $f : (D^2, S^1) \to (M, \partial M)$ such that $f|_{S^1} : S^1 \to \partial M$ is essential, then there exists an embedding g with the same property.

(essential : not homotopic to a constant map)

Equivalently

If $i_* : \pi_1(\partial M) \to \pi_1(M)$ is not injective then there exists an embedding $g : (D^2, S^1) \to (M, \partial M)$ such that $g|_{S^1} : S^1 \to \partial M$ represents a non-trivial element of $\ker i_*$.



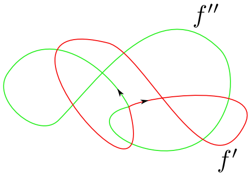
∂M
M
f
g
D^2

Baby Theorem

F a 2-manifold. If there exists an essential map $f : S^1 \to F$ then there exists an essential embedding $g : S^1 \to F$.

Proof “Cut-and-paste”

Let $s(f) = \#$ singularities of f



If $s(f) > 0$ then $f = f' * f''$
where $s(f'), s(f'') < s(f)$

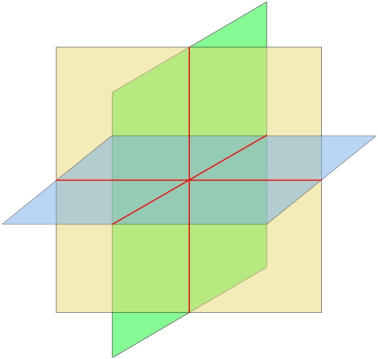
f essential $\implies$ f' or f'' essential

Done by induction on $s(f)$

Need: 2-manifolds are triangulable (Rado, 1925)

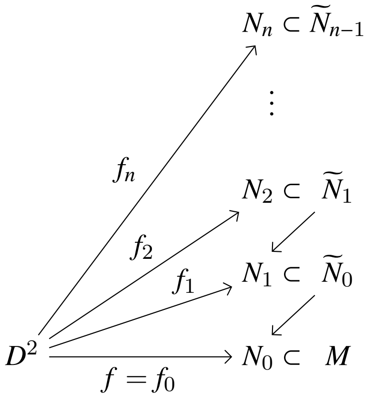
3-manifolds are triangulable (Moise, 1952)

For f : 2-manifold $\to$ 3-manifold, singularities can be more complicated; triple points:



Cut-and-paste doesn't work

Papakyriakopoulos introduced his famous tower construction



N_i = neighborhood of $f_i(D^2)$

$\widetilde{N}_i \to N_i$ covering space

∂N_n consists of 2-spheres

(1) f_n not an embedding: replace f_n by embedding g_n; gives $g : (D^2, S^1) \to (M, \partial M)$ with $s(g) < s(f)$

(2) f_n an embedding: Then f has a pair of disjoint simple double curves; cut-and-paste gives g with $s(g) < s(f)$

Disk Theorem + work of Haken (1960's) led to

Theorem (Waldhausen)

M, M' sufficiently large, prime 3-manifolds. Then

$$\pi_1(M) \cong \pi_1(M') \iff M \cong M'$$

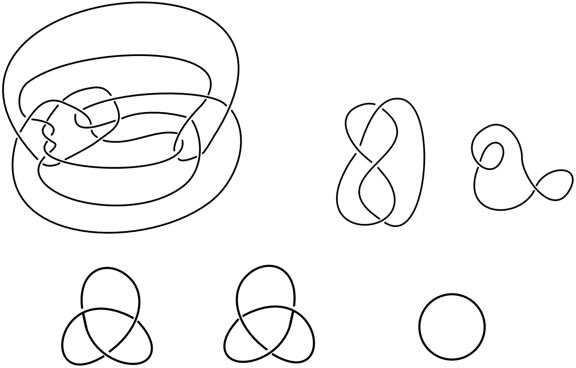
(By Perelman, can delete "sufficiently large", unless M and M' are lens spaces)

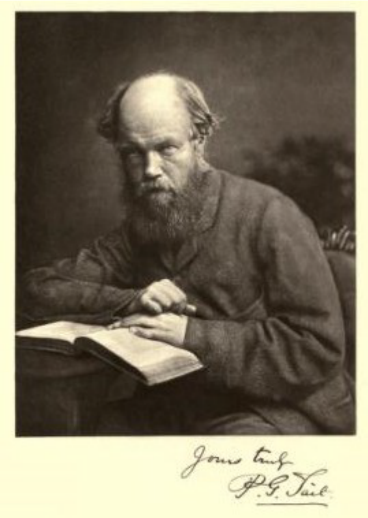
A knot is a closed loop $K \subset \mathbb{R}^3$

$K \sim K'$ if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$h(K) = K'$

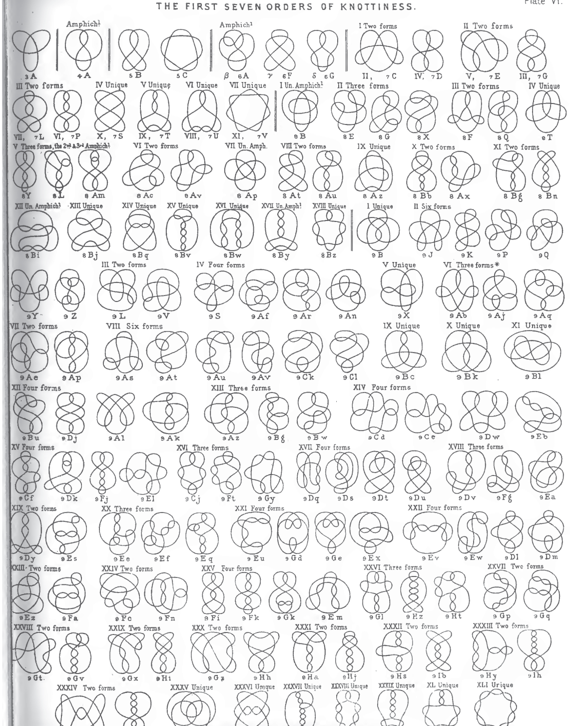
K is unknotted if $K \sim$ unit circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$





P.G. Tait (1831–1901)

Vortex Atoms


THE FIRST SEVEN ORDERS OF KNOTTINESS.
Plate VI.

$$H_1(\mathbb{R}^3 - K) \cong \mathbb{Z} \text{ for all } K$$

$$\pi(K) = \pi_1(\mathbb{R}^3 - K) = \text{the group of } K$$

$$K \text{ unknotted} \implies \pi(K) \cong \mathbb{Z}$$

Dehn's Lemma gives converse:

$$\pi(K) \cong \mathbb{Z} \iff K \text{ is unknotted}$$

Now known that

K, K' prime knots. Then
$\pi(K) \cong \pi(K') \iff K \sim K'$

Papakyriakopoulos also proved the

Sphere Theorem

M a 3-manifold. If there is an essential map $f : S^2 \to M$ (i.e. $\pi_2(M) \neq 0$) then there is an essential embedding $g : S^2 \to M$.

Cor 1 (Asphericity of Knots) $\pi_i(\mathbb{R}^3 - K) = 0, i \geq 2$.

Cor 2 (Hopf Conjecture) *U open, connected $\subset \mathbb{R}^3$. Then $\pi_1(U)$ is torsion-free.*

Cor 3 *M prime 3-manifold, not $S^1 \times S^2$, $\pi_1(M)$ infinite. Then universal cover $\widetilde{M}$ of M is contractible.*

(Now known, by Perelman, that $\widetilde{M} \cong \mathbb{R}^3$)

W. Thurston (1970's) put Poincaré Conjecture in more general context of

Geometrization Conjecture

G. Perelman proved Geometrization Conjecture (2003), using PDE's (Ricci Flow)

3-manifolds are now "classified"; in particular

The homeomorphism problem for 3-manifolds is algorithmically solvable

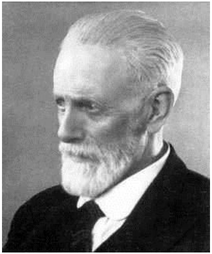
1957-1976: Papakyriakopoulos worked on Poincaré Conjecture

Tried to reduce it to a purely group-theoretic question

Achieved by J. Stallings

("How not to prove the Poincaré Conjecture" (1966))

M triangulable $\implies$ M has a Heegaard splitting

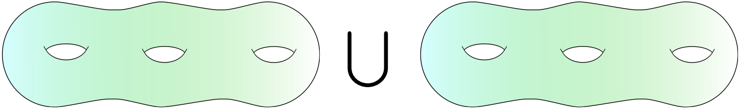


$$M = V_1 \bigcup_S V_2$$

V_i handlebody of genus g, $i = 1, 2$

$S = \partial V_1 = \partial V_2 =$ Heegaard surface

(P. Heegaard, 1895)



Get homomorphisms

$$\varphi_i : \pi_1(S) \to \pi_1(V_i) \cong F_g, \text{ free group of rank } g,\ i = 1, 2$$

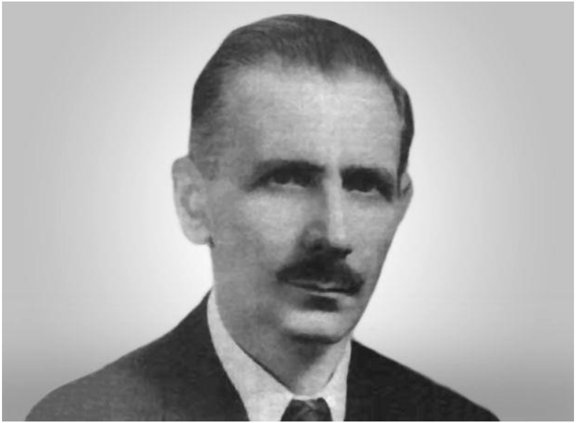
Define $\varphi : \pi_1(S) \to F_g \times F_g$ by

$$\varphi(x) = (\varphi_1(x), \varphi_2(x))$$

Easy to show $\pi_1(M) = 1 \iff \varphi$ is surjective

Theorem (Stallings + Waldhausen)

The Poincaré Conjecture is equivalent to the statement that any surjective homomorphism $\pi_1(S) \to F_g \times F_g$, S a surface of genus g, $g \geq 2$, factors non-trivially through a free product $A * B$.



Χρήστος Δημητρίου Παπακυριακόπουλος

“Papa”

June 16, 1914 – June 29, 1976